Discrete: Monatomic Chain – Classical Mechanics

$$M\ddot{u}_n = K(u_{n+1} + u_{n-1} - 2u_n)$$

$$\omega(q) = \sqrt{\frac{4K}{M}}\left|\sin\frac{qa}{2}\right|$$

$$u_n(t) = \sum_q \left(e^{iqna} \underbrace{e^{-i\omega(q)t}\frac{A_q}{\sqrt{N}}}_{B_q(t)} + e^{-iqna} \underbrace{e^{i\omega(q)t}\frac{A_q^*}{\sqrt{N}}}_{B_q^*(t)} \right) \quad \text{—— (1)}$$

$$L = \sum_{n=1}^{N} \frac{1}{2} M\dot{u}_n^2 - \sum_{n=1}^{N} \frac{1}{2} K(u_n - u_{n+1})^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_n} - \frac{\partial L}{\partial u_n} = 0 \Rightarrow M\ddot{u}_n = K(u_{n+1} + u_{n-1} - 2u_n)$$

$p_n = M\dot{u}_n$ (2) from $p_n = \dfrac{\partial L}{\partial \dot{u}_n}$

$$H = \sum_n p_n \dot{u}_n - L = \sum_n \frac{p_n^2}{2m} + \frac{K}{2}\sum_n (u_n - u_{n+1})^2 \quad \text{—— (3)}$$

Plug (1) and (2) into (3)

$$H = \sum_q M\omega^2(q)\left[B_q^*(t)B_q(t) + B_q(t)B_q^*(t)\right]$$

$$= \sum_q \frac{\hbar\omega_q}{2}\left[b_q^*(t)\, b(t) + b_q(t)\, b_q^*(t)\right] \quad \text{—— (4)}$$

where $B_q(t) \equiv \sqrt{\dfrac{\hbar}{2M\omega(q)}}\, b_q(t)$

Discrete: Monatomic chain – Quantum Mechanics

$$u_n \to \hat{u}_n \; ; \quad p_n \to \hat{p}_n$$

$$[\hat{p}_n, \hat{u}_\ell] = -i\hbar\delta_{n\ell} \, , \quad [\hat{u}_n, \hat{u}_\ell] = 0 \, , \quad [\hat{p}_n, \hat{p}_\ell] = 0$$

Since $u_\ell(t)(B_q, B_q^*)$ and $p_\ell(t)(B_q, B_q^*)$,

$B_q \to \hat{B}_q$ etc.

$$[\hat{B}_q, \hat{B}^*_{q'}] = \frac{\hbar}{2M\omega(q)} \delta_{q,q'}$$

<u>OR</u> $[\hat{b}_q, \hat{b}^+_{q'}] = \delta_{q,q'}$

This allows us to write

$$\hat{H} = \sum_q \frac{\hbar\omega(q)}{2} + \sum_q \hbar\omega(q)\, b_q^+ b_q$$

Continuum: String: Classical Field Theory

$$\rho \ddot{u}(x,t) = Ka \frac{\partial^2 u(x,t)}{\partial x^2} = g \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\omega(q) = \sqrt{\frac{g}{\rho}}\, q$$

$u(x,t)$ displacement field

$$L = \int_0^L \left[\frac{1}{2} \rho \left(\frac{\partial u(x,t)}{\partial t} \right)^2 - \frac{1}{2} g \left(\frac{\partial u(x,t)}{\partial x} \right)^2 \right] dx$$

Substitute into

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{u}(x)} - \frac{\delta L}{\delta u(x)} = 0$$

~~leads~~ needs some "new" ways to take derivatives

$$\Rightarrow \rho \ddot{u}(x,t) = g \frac{\partial^2 u(x,t)}{\partial x^2}$$

Conjugate momentum: $\pi(x) = \dfrac{\delta L}{\delta \dot{u}(x)} = \rho \dot{u}(x)$

$$H = \int \dot{u}(x) \pi(x) dx - L$$

$$= \int_0^L \left[\frac{\pi^2(x)}{2\rho} + \frac{1}{2} g \left(\frac{\partial u(x)}{\partial x} \right)^2 \right] dx$$

Continuum: QFT

Impose: $[\pi(x), u(x')] = \frac{\hbar}{i}\delta(x-x')$

$[u(x), u(x')] = 0, \quad [\pi(x), \pi(x')] = 0$

Done

At the end,

$$H = \sum_q \hbar\omega(q)\left(b_q^\dagger b_q + \frac{1}{2}\right)$$

See enrichment sections for details.